Chapter 8

What Mathematics Is

Custard

Ingredients
6 egg yolks
2 oz. superfine sugar
1 pint heavy cream, whipping cream, or milk

Method

1. Whisk the egg yolks and sugar until very thick, pale and creamy. If you watch carefully while whisking, you will see the mixture change color and get noticeably thicker, as if it's undergone a chemical change.
2. Heat the milk or cream until bubbles appear around the edge of the pan. Pour slowly into the egg mixture, stirring gently.
3. Quickly wash and dry the saucepan and pour the mixture back in. Warm it on a low heat, stirring very continuously until it coats the back of the spoon.

Making custard is thought of as a tricky process. The reason is hidden in the last step of the recipe. A more accurate description of the last step would go like this.

Watch for a thickening of the custard that looks like a qualitative change, and then take it off the heat. But don't wait until the custard is as thick as you want it, because it will continue cooking after you take it off the heat and then be overdone and probably curdle. However if you don't wait long enough then the custard will be thin and uncooked. It might help to have a glass bowl ready, with a sieve over it. Apparently if you pour the custard through a sieve it will stop cooking more quickly. I've tried it both ways and am not sure if I noticed a difference, but it does make me feel reassured that I've taken every possible precaution. If you cut it very close then the last part of the custard in the pan will be overcooked by the time you've finished pouring, so you might want to leave the last part behind.

We can now see why custard is thought of as being difficult—the instructions are not very clear-cut. It's not like measuring ingredients, setting the oven temperature, and putting on a timer. The last step requires almost an entire essay to describe it, and even then the only way to get it right is to try it plenty of times yourself. Books often say something about waiting until the custard coats the back of a wooden spoon in such a way that if you run your finger through it it leaves a mark, but I have never been able to understand this instruction, because my finger seems to leave a mark before I've even started cooking the custard mixture at all. This is one of the things I find exciting but a bit scary about making custard. You have to use your judgment, in a very short space of time, and it would be hard to get a robot to do it.

I'm now going to draw this half of the book to a close by showing that math is *easy* in the same sense that custard is difficult.

Logic vs. Illogic

Why Math Is Easy and Life Is Hard

It is a truth universally acknowledged that mathematics is difficult. Or at least so it seems, based on the number of times I tell someone I'm a mathematician only for them to respond, "Wow, you must be really clever."

This is one of the great myths of mathematics. I'm now going to take the bold step—perhaps the rash step—of exploding it. This is a bit like the Masked

Magician, whose TV show explained how magic tricks work, with the result that he was vilified by the Magical community. Nevertheless, I am going to show that mathematics is easy, and in fact that it is precisely "that which is easy."

First I'd better make clear what I mean by "easy," just as in the cake-cutting problem you first have to be clear what you mean by "fair." And here's what I mean: something is easy if it is attainable by logical thought processes. That is, without having to resort to imagination, guesswork, luck, gut feeling, convoluted interpretation, leaps of faith, blackmail, drugs, violence, and so on.

By contrast, life is hard. That is, it involves things that are not attainable by logical thought processes. This can be seen as either a temporarily necessary evil or an eternally beautiful truth. We can think either:

1. Life is like that only because we haven't yet made ourselves logically powerful enough to understand it all, and that we should be continually striving for this ultimate rational goal.

   or,

2. We will never be able to encompass everything by rationality alone, and this is a necessary and beautiful aspect of human existence.

I'm in the second camp. Here's why.

Mathematics Is Easy

As Long as You Have the Right Definition of "Easy"

What is mathematics? Earlier on I said: "Mathematics is the study of anything that obeys the rules of logic, using the rules of logic." What is mathematics for? I'll sum up the discussion of this first half of the book as follows. Math has two broad purposes:

1. To provide a language for making precise statements about concepts, and a system for making clear arguments about them.
2. To idealize concepts so that a diverse range of notions can be compared and studied simultaneously by focusing only on relevant features common to all of them.

Put more simply, mathematics is there to make difficult things easier. There are many reasons that "things" can be difficult, and mathematics doesn't deal with all of them (not directly, anyway). Here are three ways in which things can be difficult that math addresses.

1. Maybe our intuition is not strong enough to work something out.
2. Maybe there's too much ambiguity around, making it impossible to work out what's really what.
3. Maybe there are too many problems to sort out and too little time in which to do it.

Mathematics comes to our aid.

1. It helps us to construct and understand arguments that are too difficult for ordinary intuition.
2. It is a way of eliminating ambiguity so that we can know precisely what we're talking about.
3. It cuts corners, answering many questions at the same time by showing that they're all actually the same question.

How does it do it? By abstraction: throwing out the things that cause ambiguity, and ignoring any details that are irrelevant to the question in hand. You keep doing this throwing-out-and-ignoring, until you get to a point where all you have to do is apply unambiguous logical thought and nothing else.

Bananas and Blondes

Ignoring Difficult Details

Here are some problems that we might try to sort out using our techniques of math.

1. A banana and a banana and a banana is three bananas, a frog and a frog and a frog is three frogs, and so on. So we think: *Hmm, something's going on here.* And it becomes $1 + 1 + 1 = 3$.

2. What about if we say, "3 blondes and 2 brunettes is how many people?" We discard the irrelevant notion of hair color, and the question becomes: "3 people and 2 people is how many people?" And finally this becomes a sum: $3 + 2 = ?$

3. My father is twice as old as me but ten years ago he was three times as old as me. How old is he? Or: This bag has twice as many apples as that one but if I take ten out of each then this one has three times as many as that one. How many apples are there? Both of these become a pair of equations:
$$\begin{aligned} x &= 2y \\ x - 10 &= 3(y - 10). \end{aligned}$$

Now, in this case you might well have been able to do it without explicitly using simultaneous equations. But what about this problem—can you do this one in your head?

A rope over the top of a fence has the same length on each side, and weighs one-third of a pound per foot. On one end hangs a monkey holding a banana, and on the other end a weight equal to the weight of the monkey. The banana weighs 2 ounces per inch. The length of the rope in feet is the same as the age of the monkey, and the weight of the monkey in ounces is as much as the age of the monkey's mother. The combined age of the monkey and its mother is 30 years. Half the weight of the monkey plus the weight of the banana is a quarter the sum of the weights of the rope and the weight. The monkey's mother is half as old as the monkey will be when it's three times as old as its mother was when she was half as old as the monkey will be when it's as old as its mother will be when she's four times as old as the monkey was when it was twice as old as its mother was when she was a third as old as the monkey was when it was as old as its mother was when she was three times as old as the monkey was when it was a quarter as old as it is now. How long is the banana?

4. I am very happy. How will I feel if I go bungee-jumping? This has far too much ambiguity. So what does mathematics do with it? It ignores it. (Which makes it much easier.)

5. We want to understand how playing snooker works. First we imagine that everything is perfectly spherical, perfectly smooth, and perfectly rigid. We might think about relevant details like friction, bounciness, spin, and so on later. We can ignore irrelevant details like color. Except color is not irrelevant in practice; but the added pressure of trying to pot the black to win is not a question that mathematics can deal with.

This is the crucial point: we make things easy by ignoring the things that are hard. Mathematics is all the parts we don't have to throw away. The easy bits.

If Math Is Easy, Why Is It Hard?

You might be wanting to point out a flaw in my argument already: if math is easy, why does anyone find it hard? There are as many ways to make things difficult as there are to make them easy, and we can be sure that a whole ton of them have been applied to mathematics.

If someone finds math hard it might also be because nobody told them what it was for. A fork is rather hard to use as a knife. It's also rather hard to use if you're trying to eat a sandwich. Or a bowl of soup. Or a bag of popcorn.

If someone finds math hard it might also be because they have no desire to answer the question that the math is simplifying. Trigonometry makes triangles really easy. But if you don't care about triangles you're unlikely to feel that your life has been made easier by trigonometry.

But also, some people just will find things much harder if they're not allowed to use imagination, guesswork, or violence. Rationality says that this behavior is to be deplored as we head towards the ideal of ultimate rationality.

The Aim of Ultimate Rationality

Many people, especially mathematicians, philosophers, and scientists, think that we as human beings should aim to become completely rational. That if we discover a way in which we're not rational, we should get rid of it, iron it out, in order to get closer to the goal of ultimate rationality. This has two facets:

1. We should *be* completely rational (that is, behave rationally and think rationally).

2. We should be able to *understand* everything completely rationally.

I want to look at a little logic in order to work out what this might mean.

Background on Logic

There's a standard logic exam question for undergraduates that tries to show, using logic, why democracy doesn't work. This is different from Arrow's Theorem, described earlier, which shows that voting systems can't be fair. This time we're going to show that democracy doesn't work as a policy-making system.

The basic assumption we start with is that everyone in the democracy is *rational*. This is defined in terms of their beliefs: we say that their beliefs should be somehow sensible.

To make it more precise (which is what mathematicians do) we say the beliefs of any individual are "consistent" and "deductively closed." What does this mean?

A set of beliefs is called *consistent* if it doesn't imply a contradiction. For starters, this means you don't believe that something is both true and false. For example, "I am clever, I am not clever" is clearly inconsistent. But more than that, you don't believe anything that *leads to* a contradiction. For example if you believe

A. All mathematicians are clever.

B. I am a mathematician.

C. I am not clever.

This leads to a contradiction, because A and B together imply that I am clever, which contradicts C.

Your set of beliefs is called *closed* if anything you can logically deduce from your beliefs is also one of your beliefs. For example, if you believe

A. All mathematicians are clever.

B. I am a mathematician.

then you must also believe

C. I am clever.

The exam question then essentially says this. Suppose there is a vote on all beliefs, and that the government is supposed to act according to what the majority thinks on each belief. Now look at the set of "things believed by a majority of people" (not necessarily the same majority each time): is this consistent or deductively closed? The trouble is that it is neither.

Here's how this question looks when written out formally:

> The beliefs of each member i of a finite non-empty set I of individuals are represented by a consistent, deductively closed set S_i of propositional formulae. Show that the set
>
> $$\{t \mid \text{all members of } I \text{ believe } t\}$$
>
> is consistent and deductively closed. Is the set
>
> $$\{t \mid \text{over half the members of } I \text{ believe } t\}$$
>
> deductively closed or consistent?

Whether written formally or not, it's all a bit abstract, so let's pick an example. We'll use the following three statements/beliefs:

A. College education should be free.

B. Everyone should have the chance to go to college.

C. The government should spend more on college education.

Think for a moment about which of those three statements you agree with. I think you'll agree that if you think college should be free and that

everyone should have a chance to go, the government (or someone with a vast amount of money) will have to spend more on universities. That is, statements A and B together imply C (unless we allow college education to get a lot worse).

Now suppose we have a grand total of three people in this democracy. We can already produce a problem. Imagine that our three people believe the following things.

- Person 1 believes all three things.
- Person 2 believes college should be free, but the government should not spend more on college education. (To make this work, not everyone will be able to go to college.)
- Person 3 believes that everyone should be able to go to college, but the government should not spend more on it. (To make this work, college education can't be free.)

Now let's see what the majority thinks. In this case, a "majority" means at least two people.

- Two people believe that college should be free.
- Two people believe that everyone should have a chance at college.
- Two people believe the government should not spend more money on education.

Now we try to make policy based on these majority beliefs. We have a problem—we are supposed to make college free and open to everyone, without the government spending any more money on it. The majority beliefs in this case are neither consistent nor deductively closed. Oh dear.

Life Is Difficult

Life, frankly, is difficult. And in that context, this idea of a "completely rational person" is absurd.

The upshot is that rational thinking simply isn't good enough to cope with all that life throws at us. Rationality fails us in life, because:

- It's too slow.
- It's too methodical.
- It's too inflexible.
- It's too weak.
- It's too powerful.
- It has no starting point.

And that's why irrationality (or "arationality") and illogic are not human weaknesses but human *strengths* when used appropriately.

Logic Is Too Slow

In life, we don't always have time to go through logical thought processes to come to a decision. Emergency situations are much more urgent than that, and in those cases the important thing is to make a decision that is fast rather than accurate at all costs. There's no point being right if you've already been flattened by the oncoming truck.

How do we know how to throw and catch? How do we sing in tune (if we do sing in tune)? There is math behind both of those things but we don't have time to calculate trajectories or vocal cord tensions while catching or singing.

The speed issue is why we have reflex actions. We have built-in reflex actions, but we can also train reflex actions, like learning to say "You're welcome" automatically every time someone says "Thank you," or learning how to walk all the way to lectures even when you're still pretty much asleep.

Logic Is Too Methodical

Logical thought proceeds calmly step by step through logical inferences. This isn't just slow, it's boring. You don't get into uncharted territory by taking little tiny safe baby steps. Remember that game Green Light, Red Light? Someone stands at the front and turns their back. Everyone else stands some distance away and has to try and reach the front first. But the person at the front

can turn around at any moment, and if they see you moving, you're sent back to the beginning. My memory of this game is that I never won because I was too cautious; the people who won were the daring ones who took great big steps instead of tiny little ones like me.

The big leaps in life are the flashes of inspiration. These are nothing to do with logic. They happen both in mathematics and in other creative parts of life. The great geniuses of history are often the ones who've made great leaps of inspiration. Now, inspiration in mathematics doesn't mean there's something about mathematics that isn't logical—you still have to use logic to prove what you think is true—but often a flash of inspiration gives you the idea for what you think might be true in the first place.

It's like building bridges: it's hard to build a bridge across a river, but easy to cross the bridge once someone else has built it. And while you're trying to build the bridge, it's helpful to be able to fly.

Logic Is Too Inflexible

Logic is too inflexible in the face of a flexible and often rather random world. Logic is rigid and can't deal with that randomness.

Take our use of language. We assign words to things, which essentially means we're doing some random association of sounds with notions. Onomatopoeia aside, there's no logic to it at root. There may be some sense in the etymology of a word, but somewhere back in the history of the word is a random association that started the whole thing off. And we can do that because our brains have the capacity for random association. This is nothing to do with logic.

Logic Is Too Weak

Another situation where logic can't help us is if there isn't enough information. The great thing about logic is that it eliminates the use of imagination and guesswork. But this can be a bad thing too. There are an awful lot of situations in life where we don't have enough information to make a completely logical decision. Perhaps there is an unpredictable element, something ran-

dom, something we can't detect, or things we just don't know, or haven't got the time and resources to find out.

What are we to do, just not make those decisions? Instead, we do various things. We can think about probability. For example, a doctor tells us that 99 percent of these operations are successful, so we go ahead with it.

We can go instinctive: *I don't like the look of this dark alleyway, so I'll go a different way.* We can guess, like choosing lottery numbers. There's no logic there, but it makes some people exceedingly rich. We can go random ourselves, and let the dice decide.

Decision making is indisputably hard. You try and gather more and more information, but at some point your information (or your time) is going to run out, and logic is certainly not going to take you the rest of the way. It's just too weak. Now I'm not saying that you then have to make an irrational decision that actually goes against rationality, but you *are* going to have to make a non-rational or arational decision. Perhaps if something is pure logic, it doesn't count as a decision at all.

Logic Is Too Powerful

Apart from the fact that logic is too weak, logic is also too powerful. Its unforgivingly brutal power forces us into extreme positions if we take it too seriously.

For example:

> It's okay to drink half a pint of beer in an evening.
> If it's okay to drink x pints of beer then it's okay to drink x pints and 1 ml.
>
> In which case, it's okay to drink any number of pints in an evening.

The first two statements seem reasonable by themselves, but the last statement is clearly idiotic. And yet it follows logically from the first two. It appears that, in order to be rational (closed and consistent), we either have to believe that it's okay to drink any number of pints in an evening (which doesn't sound at all rational) or we have to believe that it's not okay to drink any beer at all, ever.

The problem here is the subtlety of a fine line, or a sliding scale, or gray area between the black and the white. Somehow we are able to deal with sliding scales in our heads in a way that logic can't. The power of logic is in this case its downfall. It brings me to Fuji's paradox.

Fuji's Paradox

I've named this paradox after a Japanese bond trader called Fuji who first drew my attention to it. It's a case in point that I don't think he noticed that there was a paradox at all.

It was back in the dark ages, before I realized that mathematics was easy, while bond trading is hard. So there I was, trading futures at Goldman Sachs, when this guy Fuji came along to tell us about the Japanese market. Now, Japanese interest rates were already the lowest in the world, and everyone was wondering whether they'd go any lower, even to zero. Fuji's theory was that they would never actually hit zero because then everyone would know that they couldn't go any lower, since negative interest rates would be absurd.

The thing is that Japanese interest rates go in increments of quarter percentage points, so the Bank of Japan can only change rates by multiples of that. So, I thought to myself, if Fuji's theory is right, then interest rates will not be set at 0.25% either, because then everyone will know they can't go any lower, since they can't be zero. Oh, but then they can't be 0.5% either. Nor can they be 0.75%, or 1%... which means that they can't be any percent—which means that Japan can't have interest rates.

This is clearly not true—Japan did have interest rates and still does. So what's gone wrong? (Actually, a couple of years later, Japanese interest rates really did go negative, but that's another unbelievable story.)

Unexpected Hanging

Fuji's paradox is in fact a manifestation of the "unexpected hanging" paradox.

The prisoner is told that he will be hanged sometime this week but on a day when he isn't expecting it. So he thinks to himself: *well it can't be on Sunday, because if I hadn't been hanged by Saturday then I'd know that it had to be Sunday, so I'd be expecting it. So it has to be Saturday at the latest. But then it can't be on Saturday, because if I hadn't been hanged by Friday and it can't be Sunday, then I'd know it had to be Saturday and I'd be expecting it. So it can't be Saturday... so it can't be Friday... or Thursday... or Wednesday... or Tuesday... or Monday—which means I won't be hanged!*

And then on Monday he is hanged, and he really isn't expecting it.

We can only imagine how miffed he feels, hanging there trying to work out where his logic went wrong.

Logic Has No Starting Point

My last charge against logic is that it has no starting point. If we're not going to take anything on blind faith, we're simply not going to get anywhere. You can't prove something from nothing; you can't deduce anything from nothing; you can't build a Lego construction without any blocks; there's no such thing as a free lunch. We saw Lewis Carroll's paradox that showed we would at least have to accept the rule of inference *modus ponens* on blind faith, otherwise we would never be able to infer anything from anything else. But even to infer anything from anything else, we have to have something to start with. (Having said that, I've had plenty of arguments with people, mostly mathematicians, who maintain that there's really nothing at all that they believe without justification.)

This seems to me to be an obvious and immediate flaw in the idea of the ultimately rational person. But does that mean we should immediately and completely give up?

The thing is, there's still some scope for greater and lesser rationality. For example:

- A rational person is supposed to believe that the earth is round.
- A rational person is supposed to believe that $1 + 1 = 2$.
- A rational person is not supposed to believe in ghosts.
- A rational person is not supposed to believe in psychic powers.
- Is a rational person supposed to believe in God?

Where do these "supposed"s come from? They come from society. It hasn't always been the norm to believe that the earth is round. And in some societies it is the rational norm to believe in God, while in others it is not. So in fact, rationality is a *sociological* notion. Apparently you can still be considered rational as long as all your basic beliefs come from the big bank of basic beliefs accepted by your society as "rational things to believe." If your basic beliefs are "the moon is made of soft green cheese" or "sleeping upside down is good for the elbows" or "I must kill as many people as possible," then someone will soon come and take you away.

But still, I've had arguments with people (mainly philosophers) who get very upset if something I'm defending comes down to something I believe, and I declare that I believe it "because I do." Rational people aren't supposed to do that, are they?

Well I believe it's a good thing to be aware of what you're assuming.

I repeat: I *believe* it's a good thing to be aware of what you're assuming. Whether it's a whole lot of things at the root of your belief tree, or, say, God.

Being aware of your assumptions is definitely part of the discipline of mathematics, and also part of what makes math easy—everyone has to state very clearly what their basic assumptions are. I don't think there's anything wrong with believing some things without justification—they are your axioms, from which all else grows. For example, I believe in love, but I have no justification for that. The crucial part is to be aware that these things are part of your axioms, and not to pretend you arrived at them by pure logic.

Mathematics Is Not Life

So: math is easy, life is hard, therefore math isn't life.

This doesn't mean that we shouldn't try to extend the scope of mathematics so that it includes as much as possible. We should do that, just as we should try to become "more and more rational" by continuing to work out what the inital premisses of our beliefs are. The pursuit of mathematics is the process of working out exactly what is easy, and the process of making as many things easy as possible.

But we should not feel affronted by the existence of things that can't be subsumed by mathematics: the irrational or arational, the "illogic" in life. Without that there would be no language, no communication, no poetry, no art, no fun.

Part II

Category Theory